

Curve Counting à la Göttsche

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Let n_δ be the number of δ -nodal curves lying in a suitably ample complete linear system $|L|$ and passing through appropriately many points on a smooth projective complex algebraic surface. Often n_δ is referred to as a *Severi degree*. A major problem is to understand the behavior of n_δ , specifically to finish off Lothar Göttsche's mostly proved 1997 conjectures [17] and then go on to treat the new refinements by Göttsche and Vivek Shende [18], [19].

The general area has been very active for over fifteen years, and is now busier and more exciting than ever before. Among many other people involved have been Joe Harris himself, some of his students, and some of theirs. The area is unusually broad—embracing ideas from physics, symplectic differential geometry, complex analytic geometry, algebraic geometry, tropical geometry, and combinatorics.

Problem number one is to find the two power series

$$B_1(q), B_2(q) \in \mathbb{Z}[[q]]$$

appearing in Göttsche's remarkable formula for the generating function of the n_δ . The formula expresses the function, so the n_δ , in terms of the four basic numerical invariants of the system and the surface. In fact, n_δ is a polynomial in the four. See (1) and (4) and (5) below.

Göttsche [17, Rmk. 2.5(2)] computed the coefficients of $B_1(q)$ and $B_2(q)$ up to degree 28 on the basis of the recursive formula for the n_δ of the plane due to Lucia Caporaso and Harris [8, Thm. 1.1]. (A different recursion had been given earlier by Ziv Ran [34, Thm. 3C.1].) Göttsche checked the result against much of what was known, including Ravi Vakil's enumeration [40] for the Hirzebruch surfaces (that is, the rational ruled surfaces).

The **problem** is to find a closed form for each $B_i(q)$, or a functional equation.

Second, given δ , how ample is suitable, so that n_δ has the predicted value? After all, for any system, the polynomial yields a number, but it isn't always n_δ . For example, consider plane curves of degree d . If $d = 1$, then n_3 is the number of 3-nodal lines, namely 0, but the polynomial yields 75. Considering the geometry, Göttsche [17, Cnj. 4.1, Rmk. 4.4] conjectured the polynomial works if $d \geq \delta/2 + 1$.

The latter conjecture was proved for $\delta \leq 8$ by Ragni Piene and the author [25,

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Thm. 3.1] using algebraic methods, then for $\delta \leq 14$ by Florian Block [4, Prp. 1.4]. He built on ideas of Sergey Fomin and Grigory Mikhalkin [15, Thm. 5.1], who used tropical methods to set up the enumeration from scratch and to validate its predictions for $d \geq 2\delta$. In principle, the problem is purely combinatorial: to show formally the Caporaso–Harris recursion yields a polynomial in d for $d \geq \delta/2 + 1$.

On any surface, Martijn Kool, Shende, and Richard Thomas [27, Prp. 2.1] proved it suffices for L to be δ -very ample. Piene and the author [24, Thm. 1.1] proved it suffices for L to be of the form $M^{\otimes m} \otimes N$ where M is very ample, $m \geq 3\delta$, and N is spanned, provided $\delta \leq 8$. Both results were inspired by Göttsche’s [17, Prp. 5.2]; in turn, Göttsche had been inspired by Harris and Rahul Pandharipande’s paper [20], which treats the case $\delta \leq 3$ in the plane.

In fact, Göttsche conjectured the polynomial works for plane curves of degree d iff $d \geq \delta/2 + 1$. And Block proved, for $3 \leq \delta \leq 14$, that $\lceil \delta/2 \rceil + 1$ is, indeed, a *threshold*, as he called it; namely, it is the least integer d^* such that the polynomial works for $d \geq d^*$. Further, Göttsche conjectured a similar statement for the Hirzebruch surfaces. Shende and the author [26] proved that, above Göttsche’s conjectured threshold, the polynomials work for the plane and for the Hirzebruch surfaces and that a similar statement holds for the classical del Pezzo surfaces; moreover, there’s at least one case where the polynomial works below the conjectured threshold too.

Sometimes, the curves are required to belong to a general linear subsystem of $|L|$ rather than to pass through appropriately many points. However, the latter condition does yield a general subsystem by Piene and the author’s [24, Lem. (4.7)].

The **problem** is to determine just when the polynomial yields n_δ .

Third, what about nonlinear systems? After all, Gromov–Witten theory fixes not the linear equivalence class, but the homology class, and this class determines the four basic invariants, (1) below. Jim Bryan and Naichung Conan Leung [5, Thm. 1.1] handled primitive complete nonlinear systems on generic Abelian surfaces for all δ . They used symplectic methods. Piene and the author [25, § 5] obtained similar results algebraically, but for $\delta \leq 8$.

Israel Vainsencher [39, § 6.2] treated a remarkable system. His parameter space was the Grassmannian of \mathbb{P}^2 in \mathbb{P}^4 . His surface was \mathbb{P}^2 , but moving in \mathbb{P}^4 . His curves arose by intersecting the moving \mathbb{P}^2 with a fixed general quintic 3-fold X . Thus he found X contains 17,601,000 irreducible 6-nodal quintic plane curves. Piene and the author [25, Thm. 4.3] validated the number. Pandharipande [11, (7.54)] noted each curve has six double covers previously unconsidered in mirror symmetry.

Given any suitably general algebraic system of curves on surfaces, Piene and the author [25, Thm. 2.5 and Rmk. 2.7] found on the parameter space the class of the curves with δ nodes for $\delta \leq 8$ and conjectured the formula generalizes to any δ .

The **problem** is to generalize the formula for n_δ , in (4), to algebraic systems.

Fourth, what about higher singularities? This question is related to the previous one, about algebraic systems. For example, given a system, consider those curves with a triple point and δ double points. Their number can be viewed as the number of curves with δ double points in the following system: take the subsystem of curves with a triple point, and resolve the locus of triple points. This example was treated for $0 \leq \delta \leq 3$ by Vainsencher and by Piene and the author [24, Thm. 1.2]. A substantial amount of work has been done; see Maxim Kazarian’s paper [21], Dmitry Kerner’s papers [22], [23], Jun Li and Yu-Jong Tzeng’s paper [28], Jørgen Rennemo’s paper [35] and their references.

The **problem** is to enumerate the curves of fixed global equisingularity type lying in a given algebraic system — that is, to find on the parameter space the class of these curves.

Fifth, what about positive characteristic? Sometimes an enumeration is more tractable modulo a prime. Thus Göttsche [16, Thm. 0.1] found the Betti numbers of the Hilbert schemes of points on a smooth surface. (In [13, pp. 175–178], he and Barbara Fantechi discuss other proofs and refinements of the result.) This result, combined with others, led to the celebrated formula of Shing-Tung Yau and Eric Zaslow [41, p. 5] enumerating rational curves on a K3 surface. They developed ideas of Cumrun Vafa et al.: see [37, p. 438] for a similar formula; see [38, p. 44] for the use of Göttsche’s result; see [3, p. 437] for the use of varying Jacobians. In turn, Arnaud Beauville [1] and Fantechi, Göttsche, and Duco van Straten [14] developed the ideas in [41] further, and Xi Chen [9, Thm. 1] proved the curves are nodal.

The Yau–Zaslow formula too inspired Göttsche to develop his conjectures. For K3 surfaces and Abelian surfaces, $B_1(q)$ and $B_2(q)$ disappear, leaving explicit formulas in any geometric genus. These formulas were proved for primitive classes on generic such surfaces by Bryan and Leung; see [6] for a lovely survey.

The **problem** is to determine just when Göttsche’s conjectures hold in positive characteristic.

To define the $B_i(q)$, denote the surface by S , and its canonical bundle by K . The four basic invariants are these numbers:

$$(1) \quad x := L^2, \quad y := L \cdot K, \quad z := K^2, \quad t := c_2(S).$$

For $\delta \leq 6$, Vainsencher [39, § 5] worked out formulas for the n_δ , getting enormous polynomials in x, y, z, t . Afterwards, it was natural to conjecture this statement:

$$(2) \quad \text{The number } n_\delta \text{ is given by a universal polynomial of degree } \delta \text{ in } \mathbb{Q}[x, y, z, t].$$

For plane curves of degree d , we have $(x, y, z, t) = (d^2, -3d, 9, 3)$. So Philippe Di Francesco and Claude Itzykson [12, p. 85] conjectured n_δ is given by a polynomial in d of a certain shape for $\binom{d-1}{2} \geq \delta$. Youngook Choi [10, p. 12] established their conjecture for $d \geq \delta$ on the basis of Ran’s work [34]. Göttsche [17, § 4] refined the conjecture. Given (2) in the form of (4) below, Nikolay Qviller [33, § 4] established most of Göttsche’s refinements concerning the shape.

In full generality, (2) was given a symplectic proof and an algebraic proof by Ai-ko Liu [29], [30]. It was recently given new proofs by Tzeng [36, Thm. 1.1] and Kool, Shende, and Thomas [27, Thm. 4.1]; the former is purely algebraic, whereas the latter also relies on topology. These new proofs have caused quite a stir!

Göttsche [17, Cnj. 2.1] did conjecture (2) in full generality, but his elaboration is far more important. First, he proved (2) is equivalent to this statement:

$$(3) \quad \sum n_\delta u^\delta = A_1^x A_2^y A_3^z A_4^t \quad \text{for some } A_i \in \mathbb{Q}[[u]].$$

The A_i are the exponentials of their logarithms. Hence (3) is equivalent to this:

$$(4) \quad n_\delta = P_\delta(a_1, \dots, a_\delta) / \delta! \quad \text{where } \sum_{\delta \geq 0} P_\delta u^\delta / \delta! = \exp(\sum_{\kappa \geq 1} a_\kappa u^\kappa / \kappa!)$$

for some *linear* forms $a_\kappa(x, y, z, t)$. The polynomials P_δ were studied extensively in 1934 by Eric Temple Bell [2]. Piene and the author [24, p. 210] determined a_κ for $\kappa \leq 8$, and found its coefficients to be integers. Recently, Qviller [33, Thm. 2.4] (see [32, § 6] too) proved the coefficients are always integers.

The $B_i(q)$ appear in the next formula, the *Göttsche–Yau–Zaslow Formula*:

$$(5) \quad \sum n_\delta u(q)^\delta = B_1(q)^z B_2(q)^y B_3(q)^x B_4(q)^{-\nu/2}$$

where $u(q)$, $B_3(q)$, $B_4(q) \in \mathbb{Z}[[q]]$ are explicit quasi-modular forms and where

$$\chi := \chi(L) = (x - y)/2 + \nu \quad \text{and} \quad \nu := \chi(\mathcal{O}_S) = (z + t)/12.$$

Göttsche [17, Cnj. 2.4] conjectured (5). He [17, Rmks. 2.5(1), 3.1] noted (5) implies (2) and generalizes the Yau–Zaslow Formula. Tzeng [36, Thm. 1.2] derived (5) from (3) via Bryan and Leung’s work on K3 surfaces [7, Thm. 1.1] and via Pienne and the author’s [25, Lem. 5.3]; the latter provides enough suitably ample primitive classes.

Finally, Göttsche and Shende were inspired by Kool, Shende, and Thomas’s work to conjecture, *among many other statements*, refinements [19, Cnj. 75] of the Caporaso–Harris and Vakil recursions. Further, Göttsche and Shende [19, Cnj. 5, 7] refine (5) with this conjecture: there should be polynomials $n_\delta(v) \in \mathbb{Z}[v]$ and power series with polynomial coefficients $u(v, q)$, $B_i(v, q) \in \mathbb{Q}[v, v^{-1}][[q]]$ such that

$$\sum n_\delta(v) u(v, q)^\delta = B_1(v, q)^z B_2(v, q)^y B_3(v, q)^x B_4(v, q)^{-\nu/2}$$

and such that putting $v = 1$ recovers (5). Again $u(v, q)$ and $B_3(v, q)$ and $B_4(v, q)$ are known; however, it is an **open problem** to find the geometric meaning of $n_\delta(v)$.

If S is a real toric variety, then $n_\delta(-1)$ is conjectured in [19, Cnj. 90] to be the tropical Welschinger invariant—the number of real δ -nodal curves lying in a suitably ample real complete linear system and passing through a subtropical set of appropriately many real points, each curve counted with an appropriate sign. The notion of subtropical set was introduced and studied by Mikhalkin in [31]. This conjecture is also stated by Block and Göttsche in a paper currently being written; further, there the conjecture is proved for $\delta \leq 8$ using methods like those in [4]

The **refined problem number one** is to find $B_1(v, q)$ and $B_2(v, q)$.

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